Exercises for Stochastic Processes

Tutorial exercises:

T1. Let $S = \{0, 1\}$. Consider the general Q-matrix

$$\begin{pmatrix} -\beta & \beta \\ \delta & -\delta \end{pmatrix}, \tag{1}$$

for some $\beta, \delta \geq 0$. Show that the corresponding transition probabilities are $p_t(x, y) = \mathbb{1}_{\{x=y\}}$ if $\beta + \delta = 0$, and otherwise they are given by

$$\begin{aligned} p_t(0,0) &= \frac{\delta}{\beta+\delta} + \frac{\beta}{\beta+\delta} e^{-t(\beta+\delta)}, \quad p_t(0,1) = \frac{\beta}{\beta+\delta}\left(1 - e^{-t(\beta+\delta)}\right), \\ p_t(1,1) &= \frac{\beta}{\beta+\delta} + \frac{\delta}{\beta+\delta} e^{-t(\beta+\delta)}, \quad p_t(1,0) = \frac{\delta}{\beta+\delta}\left(1 - e^{-t(\beta+\delta)}\right). \end{aligned} \tag{2}$$

T2. Consider the following stochastic process $X(t)$ on $\{0, 1\}$. If the process is in 0 it stays in this state for an exponential distributed time with parameter β and then jumps to state 1. If the process is in state 1 it stays in this state for an exponential distributed time with parameter δ and the goes to 0. Let $p_t(i, j)$ be the probability that $X(t) = j$ if $X(0) = i$.

(a) Show that

$$p_t(0,1) = \int_0^t \beta e^{-\beta s} p_{t-s}(1,1) \mathrm{d}s,$$

and

$$p_t(1,0) = \int_0^t \delta e^{-\delta s} p_{t-s}(0,0) \mathrm{d}s,$$

(b) Show that the Q-matrix for this process is the same as in (1), so that the transition probabilities for this process are given in (2).

T3. With the notations used in the lecture in the probabilistic construction of a Markov chain with a given Q-matrix, show that the following statements are equivalent:

(a) $\mathbb{P}(N(t) < \infty) = 1$ for all $t \geq 0$.

(b) $\sum \tau_n = \infty$ a.s.

(c) $\sum \frac{1}{c(Z_n)} = \infty$ a.s.

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Homework exercises:

H1. Let Q be a Q-matrix on a finite state space. Show that $p_t(x,y)$ defined by

$$P_t := \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!}$$

is a transition function and show that

$$q(x,y) = \frac{\mathrm{d}}{\mathrm{d}t} p_t(x,y)\bigg|_{t=0} .$$

H2. Let (X_n) be a sequence of independent continuous time Markov chains on $\{0,1\}$ with Q-matrices $\begin{pmatrix} -\beta_n & \beta_n \\ \delta_n & -\delta_n \end{pmatrix}$. Assume that $\sum \frac{\beta_n}{\beta_n+\delta_n} < \infty$. Define

$$X(t) := (X_1(t), X_2(t), \dots)$$

and

$$S := \left\{ x \in \{0,1\}^{\mathbb{N}} \;\middle|\; \sum x_n < \infty \right\} .$$

(a) Show that S is countable and $\mathbb{P}(X(t) \in S \mid X(0) \in S) = 1$.

(b) Show that $p_t(x,y) := \mathbb{P}(X(t) = y \mid X(0) = x)$ is a transition function on S.

(c) Assume that, moreover, $\sum \beta_n = \infty$. Show that $c(x) = \infty$ for all $x \in S$.

(d) Show that, for any $x \in S$ and $\epsilon > 0$,

$$\mathbb{P}^x(X(t) = x \text{ for all } t < \epsilon) = 0 .$$

Conclude that there is no Markov chain with transition function p.

H3. Compute the transition function for the "linear birth chain", i.e. the (unique) continuous time Markov chain on $\mathbb{N}_0$ with Q-matrix given by

$$q(m,n) := \begin{cases} \rho m & \text{if } n = m+1, \\ -\rho m & \text{if } n = m, \end{cases}$$

(where $\rho > 0$ is an intensity parameter.)

Deadline: Monday, 02.12.19